# DIRECTIONAL DERIVATIVES

Derivative of a scalar function in the direction of a vector $\vec{a}$

$$\boxed{\frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|}\cdot(\text{grad}\, f)}$$

## EXERCISE # 5.4

Find the derivative of f at p in direction of $\vec{a}$, where

Q1 / 77 $f = e^{x+y+z}$, $p(0,0,0)$, $\vec{a} = \vec{i} + 2\vec{j} - 2\vec{k}$

Sol:- Directional derivative of f

$$\frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|}\cdot \text{grad}\, f \quad \text{———} \; (1)$$

$$f = e^{x+y+z}$$

$$\frac{\partial f}{\partial x} = e^{x+y+z}\,.\,1 = e^{0+0+0} = e^{0} = 1$$

$$\frac{\partial f}{\partial y} = e^{x+y+z}\,.\,1 = e^{0+0+0} = e^{0} = 1$$

$$\frac{\partial f}{\partial z} = e^{x+y+z}\,.\,1 = e^{0+0+0} = 1$$

$$\text{grad } f = \vec{\nabla} f = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] = [1,1,1]$$

$$① \Rightarrow \quad \frac{df}{ds} = \frac{\vec{i} + 2\vec{j} - 2\vec{k}}{|\vec{i} + 2\vec{j} - 2\vec{k}|} \cdot [1,1,1]$$

$$= \frac{[1, 2, -2]}{\sqrt{1+4+4}} \cdot [1,1,1]$$

$$= \frac{1(1) + 2(1) + (-2)(1)}{3}$$

$$\boxed{\frac{df}{ds} = \frac{1}{3}} \quad \text{Ans}$$

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Q2/77 $\quad f = e^{yz}\cos x + e^{zx}\cos y + e^{xy}\cos z$

$P(\pi/6, \pi/3, 0)$, $\vec{a} = 3\vec{i} + 2\vec{j} - \vec{k}$

Sol:- Directional derivative of $f$

$$\frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot \text{grad } f \quad \text{———} ①$$

$$f = e^{yz}\cos x + e^{zx}\cos y + e^{xy}\cos z$$

$$\frac{\partial f}{\partial x} = e^{yz}(-\sin x) + \cos y \, e^{zx}(\cdot z) + \cos z \, e^{xy} \cdot y$$

$$= e^{\pi/3 \cdot 0}(-\sin \pi/6) + \cos \pi/3 \left(e^{0 \cdot \pi/6} \cdot 0\right) + \cos 0 \, e^{\pi/6 \cdot \pi/3} \cdot \pi/3$$

$$= 1(-1/2) + \left(\frac{1}{2}\right)(0) + 1 \cdot e^{\pi^2/18} \cdot \pi/3$$

$$\boxed{\frac{\partial f}{\partial x} = -\frac{1}{2} + \frac{\pi}{3} e^{\frac{\pi^2}{18}}}$$

$$\frac{\partial f}{\partial y} = \left(e^{yz} \cdot z\right) \cos x + e^{zx}\left(-\sin y\right) + \left(e^{xy} \cdot x\right) \cos z$$

$$= \left(e^{\pi/3 \cdot 0} \cdot 0\right) \cos \pi/6 + e^{0 \cdot \pi/6}\left(-\sin \pi/3\right) + \left(e^{\frac{\pi}{6} \cdot \frac{\pi}{3}} \cdot \frac{\pi}{6}\right) \cos 0$$

$$= 1\left(-\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6} e^{\frac{\pi^2}{18}}$$

$$\boxed{\frac{\partial f}{\partial y} = \frac{-\sqrt{3}}{2} + \frac{\pi}{6} e^{\pi^2/18}}$$

$$\frac{\partial f}{\partial z} = \left(e^{yz} \cdot y\right) \cos x + \left(e^{zx} \cdot x\right) \cos y + e^{xy}\left(-\sin z\right)$$

$$= \left(e^{\pi/3 \cdot 0} \cdot \pi/3\right) \cos \pi/6 + \left(e^{0 \cdot \frac{\pi}{6}} \cdot \frac{\pi}{6}\right) \cos \frac{\pi}{3} + e^{\frac{\pi}{6} \cdot \frac{\pi}{3}}\left(-\sin 0\right)$$

$$= \frac{\pi}{3}\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\left(\frac{1}{2}\right)$$

$$\boxed{\frac{\partial f}{\partial z} = \frac{\pi}{2\sqrt{3}} + \frac{\pi}{12}}$$

$$\text{grad } f = \vec{\nabla} f = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] f$$

$$= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

$$= \left[-\frac{1}{2} + \frac{\pi}{3} e^{\pi^2/18}, -\frac{\sqrt{3}}{2} + \frac{\pi}{6} e^{\pi^2/18}, \frac{\pi}{2\sqrt{3}} + \frac{\pi}{12}\right]$$

$$= \frac{[3, 2, -1]}{\sqrt{3^2 + 2^2 + 1}} \cdot \left[-\frac{1}{2} + \frac{\pi}{3} e^{\pi^2/18}, \frac{-\sqrt{3}}{2}, \frac{\pi}{2\sqrt{3}} + \frac{\pi}{12}\right]$$

$$= \frac{1}{\sqrt{14}}\left[-\frac{3}{2} + \pi e^{\pi^2/18} - \sqrt{3} + \frac{\pi}{3} e^{\pi^2/18} - \frac{\pi}{2\sqrt{3}} - \frac{\pi}{12}\right]$$

$$\frac{df}{ds} = \frac{1}{\sqrt{14}}\left[\frac{-3}{2} - \sqrt{3} + \pi e^{\pi^2/18} + \frac{\pi}{3} e^{\pi^2/18} - \pi\left(\frac{1}{2\sqrt{3}} + \frac{1}{12}\right)\right]$$

Ans

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Q3/78 $\quad f = \frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}, \quad P(1,-1,1)$

$\vec{a} = \vec{i} - \vec{j} + \vec{K}$

Sol:- Directional derivative is

$$\frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot (\text{grad } f) \quad \text{—— (1)}$$

$$f = \frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}$$

$$\frac{\partial f}{\partial x} = \frac{-yz}{x^2} + \frac{z}{y} + \frac{y}{z} = \frac{-(-1)(1)}{1^2} + \frac{1}{-1} + \frac{-1}{1}$$

$$\boxed{\frac{\partial f}{\partial x} = -1}$$

$$\frac{\partial f}{\partial y} = \frac{z}{x} - \frac{zx}{y^2} + \frac{x}{z} = \frac{1}{1} - \frac{1\cdot 1}{(-1)^2} + \frac{1}{1}$$

$$\boxed{\frac{\partial f}{\partial y} = 1}$$

$$\frac{\partial f}{\partial z} = \frac{y}{x} + \frac{x}{y} - \frac{xy}{z^2} = \frac{-1}{1} + \frac{1}{-1} - \frac{1(-1)}{1^2}$$

$$\boxed{\frac{\partial f}{\partial z} = -1}$$

$$\text{grad } f = \vec{\nabla} f = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

$$\text{grad } f = [-1, 1, -1]$$

① ⟹

$$\frac{df}{ds} = \frac{[1,-1,1]}{\sqrt{1^2+1^2+1^2}} \cdot [-1,1,-1]$$

$$= \frac{1(-1)+(-1)(1)+1(-1)}{\sqrt{3}} = \frac{-3}{\sqrt{3}} = \frac{-\sqrt{3}\sqrt{3}}{\sqrt{3}}$$

$$\boxed{\frac{df}{ds} = -\sqrt{3}}$$ Ans

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Q4/78 $f = x^2+y^2+z^2$, $p(2,2,2)$, $\vec{a} = \vec{i}+2\vec{j}-3\vec{k}$

Sol:- $\frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot \text{grad } f$ —— ①

$f = x^2+y^2+z^2$

$\frac{\partial f}{\partial x} = 2x+0+0 = 2(2) = 4$

$\frac{\partial f}{\partial y} = 0+2y+0 = 2(2) = 4$

$\frac{\partial f}{\partial z} = 0+0+2z = 2(2) = 4$

$\text{grad } f = \vec{\nabla} f = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}] f = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}]$

$\text{grad } f = [4,4,4]$

So $\frac{df}{ds} = \frac{\vec{i}+2\vec{j}-3\vec{k}}{\sqrt{1^2+2^2+(-3)^2}} \cdot \text{grad } f$

$$= \frac{(\vec{i}+2\vec{j}-3\vec{k}) \cdot [4,4,4]}{\sqrt{1+4+9}} = \frac{4\vec{i}+8\vec{j}-12\vec{k}}{\sqrt{14}}$$

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Q6/78 $f = x^2yz + 4yz^2$, $P(1,-2,1)$, $\vec{a} = 2\vec{i}-\vec{j}-2\vec{k}$

Sol:-

$\frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot \text{grad } f$ ⟶ ①

(Q11 / 78) Find the rate of change of $f = x^2 + y z$ at $(3, 1, -5)$ in the direction of $\vec{i} + 2\vec{j} - \vec{k}$.

Sol:- Here $f = x^2 + yz$ , $P(3,1,-5)$, $\vec{a} = \vec{i} + 2\vec{j} - \vec{k}$

Rate of change of $f$ in the direction of vector $\vec{a}$ is directional derivative.

$$\frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot \text{grad } f \quad \text{— (1)}$$

$$f = x^2 + yz$$

$$\frac{\partial f}{\partial x} = 2x = 2(3) = 6$$

$$\frac{\partial f}{\partial y} = z = -5$$

P.T.O





$$\frac{\partial f}{\partial z} = y = 1$$

$$\text{grad } f = \vec{\nabla} f = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] f$$

$$= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = [6, -5, 1]$$

(1) $\Rightarrow$

$$\frac{df}{ds} = \frac{[1, 2, -1]}{\sqrt{1^2 + 2^2 + (-1)^2}} \cdot [6, -5, 1]$$

$$= \frac{(1)(6) + (2)(-5) + (-1)(1)}{\sqrt{1+4+1}} = \frac{6-10-1}{\sqrt{6}}$$

$$\boxed{\frac{df}{ds} = \frac{-5}{\sqrt{6}}} \text{ Ans}$$

(Ex-1 / 76) Find the directional derivative of the $f = x^2 + yz$ in the direction of $\vec{i} + 2\vec{j} - \vec{k}$ at the point $P(3, 1, -5)$.

Sol:- Since $\frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot \text{grad } f$ — (1)

$$\text{grad } f = \vec{\nabla} f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

$$\frac{\partial f}{\partial x} = 2x + 0 = 2(3) = 6$$

$$\frac{\partial f}{\partial y} = z = -5$$

$$\frac{\partial f}{\partial z} = 0 + y = 1$$

$$\frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot \text{grad } f$$

$$= \frac{\vec{i} + 2\vec{j} - \vec{k}}{\sqrt{(1)^2 + (2)^2 + (-1)^2}} \cdot (6, -5, 1)$$

$$= \frac{\vec{i} + 2\vec{j} - \vec{k}}{\sqrt{1+4+1}} \cdot (6, -5, 1)$$

$$= \frac{6\vec{i} - 10\vec{j} - 1\vec{k}}{\sqrt{6}}$$ Ans

let us suppose that $f(x,y,z)$ denote a Scalar function; Then for a constant $c$, the set of points satisfying the equation

$f(x,y,z) = c$

is called a level surface of $f$.

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Unit normal vector to a level Surface at the point $P$.

$$\hat{n} = \frac{(\text{grad } f)_P}{|\text{grad} f|_P}$$

where $(\text{grad } f)_P$ is called the given point.

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Find the unit normal vector to the given level surface at the given point $P$.

(Q7/78) $x^3 - y^3 + z^3 - 3xyz = 14, \quad P(2,1,-1)$

Sol:- Here $f = x^3 - y^3 + z^3 - 3xyz$

$$\frac{\partial f}{\partial x} = 3x^2 - 0 + 0 - 3yz = 3(2)^2 - 3(1)(-1) = 15$$

$$\frac{\partial f}{\partial y} = 0 - 3y^2 + 0 - 3xz = -3(1)^2 - 3(2)(-1) = 3$$

$$\frac{\partial f}{\partial z} = 0 - 0 + 3z^2 - 3xy = 3(-1)^2 - 3 \cdot 2 \cdot 1 = -3$$

$$(\text{grad } f)_P = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]_P = [15, 3, -3] = 3[5,1,-1]$$

Unit ~~vector~~ normal vector to level surface at pt $P$.

$$\hat{n} = \frac{(\text{grad} f)_P}{(|\text{grad} f|)_P} = \frac{3[5,1,-1]}{3\sqrt{5^2+1^2+(-1)^2}} = \frac{3(5\vec{i}+\vec{j}-\vec{k})}{3\sqrt{27}}$$

$$\boxed{\hat{n} = \frac{5\vec{i}+\vec{j}-\vec{k}}{3\sqrt{3}}}$$ Ans

Q8 / 78 $\quad x + y + z = 1$ , $P(4, 2, -5)$

Sol:- $f = x + y + z$

$$\frac{\partial f}{\partial x} = 1 , \quad \frac{\partial f}{\partial y} = 1, \quad \frac{\partial f}{\partial z} = 1$$

$$(\text{grad } f)_P = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]_P = [1, 1, 1]$$

$$|\text{grad } f| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

Unit normal vector to the level surface at point $P$.

$$\hat{n} = \frac{(\text{grad})_P}{(|\text{grad } f|)_P} = \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}} \quad \underline{\text{Ans}}$$

Ex-3 / 77  Find unit normal vector at the point $Q(2,1,-1)$ to the level surface.

$$x^3+y^3+z^3-3xyz = 14.$$

Sol:- $\varphi$ $f(x,y,z) = x^3+y^3+z^3-3xyz$

$$\hat{n} = \frac{(\text{grad} f)_P}{|\text{grad}|_P} \quad \text{——— (1)}$$

So, $\text{grad } f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$

$$\frac{\partial f}{\partial x} = 3x^2-3yz = 3(2)^2-3(1)(-1) = 15$$

$$\frac{\partial f}{\partial y} = 3y^2-3xz = 3(1)^2-3(2)(-1) = 9$$



$$\frac{\partial f}{\partial z} = 3z^2-3xy = 3(-1)^2-3(2)(1) = -3$$

$$\hat{n} = \frac{15\vec{i}+9\vec{j}-3\vec{k}}{\sqrt{(15)^2+(9)^2+(-3)^2}} = \frac{3(5\vec{i}+3\vec{j}-\vec{k})}{\sqrt{9\times 45}}$$

$$\hat{n} = \frac{5\vec{i}+3\vec{j}-\vec{k}}{3\sqrt{5}} \quad \text{Ans}$$

Q9 / 78  $yz+zx+xy = 5$, $P(-1,1,1)$, $\hat{n} = ?$

Sol:- $f(x,y,z) = yz+zx+xy$

$$\frac{\partial f}{\partial x} = z+y = 1+1 = 2$$

$$\frac{\partial f}{\partial y} = z+x = 1-1 = 0$$

$$\frac{\partial f}{\partial z} = y+x = 1-1 = 0$$

$$\hat{n} = \frac{2\vec{i}+0\vec{j}+0\vec{k}}{\sqrt{2^2+0^2+0^2}} = \frac{2\vec{i}}{2}$$

$$\hat{n} = \vec{i} \quad \text{Ans}$$

Q10 / 78  $x^2+y^2+z^2 = 6$, $P(1,1,1)$

Sol:- $f = x^2+y^2+z^2$

$$\frac{\partial f}{\partial x} = 2x = 2(1) = 2$$

$$\frac{\partial f}{\partial y} = 2y = 2(1) = 2$$

$$\frac{\partial f}{\partial z} = 2z = 2(1) = 2$$